Bottleneck Lower Bound

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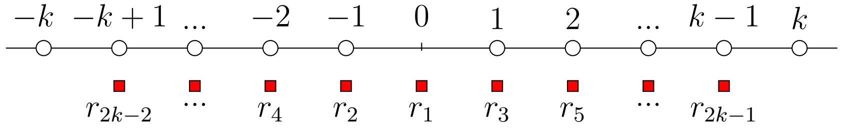
Theorem 1. The lowerbound of the competitive ratio of any randomized algorithm for the MBMBM problem with $2k$ servers is no better than $O(k)$.

Proof. In this instance the servers are located at $\{-k, -k+1, ..., -1, 1, ..., k-1, k\}$ and the requests will arrive at $(0, -1, 1, -2, 2, ..., -k+1, k-1)$ with the final request r_{2k} landing either on $-k$ or k with uniform probability.


The Instance

The important thing to note here is that after r_{2k-1} has been matched by any algorithm, there will always be exactly one server remaining, which we will denote by $s \in \mathbb{Z}$. The expected cost of the bottleneck edge in this algorithm will be at least the expected cost of the last edge created with s, since the maximal distance pairing will necessarily be bigger than or equal to $d(r_{2k}, s) = |r_{2k} - s|$. Since $s \geq -k$, the edge cost if $r_{2k} = -k$ will be $k + s$. If $r_{2k} = k$, then the edge cost will be $k - s$, since $s \leq k$. Hence, the expected cost of any algorithm will be at least $\frac{1}{2}\left((k+s) + (k-s)\right) = k$.

The offline optimal matching, however, would have a maximal pairing cost of 1. To achieve this, simply match the left most server with the left most request and repeat. If $r_{2k} = k$, then all the nonpositive requests will have a distance of 1 to their respective server while all the postive requests will have a cost of 0. The competitive ratio ρ of any algorithm ALG is defined to be

$$\rho = \max_{\text{Instances } I} \left(\frac{\text{ALG}[I]}{\text{OPT}[I]} \right).$$

Thus the competitive ratio of any algorithm will certainly be better than the ratio on this specific instance, and will thus be no better than $\frac{k}{1} = k$, where k is half the number of servers used. $\square$